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A Representation Theorem for Aumann Integrals*

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1. INTRODUCTION

In recent years the study of set-valued functions has been developed extensively by many authors, with applications to mathematical economics and control theory; see Refs. [5, 11, 15, 16]. In those papers, three approaches can be distinguished according to whether the range space (values of set-valued functions) is $\mathscr{R}^p$, a Banach space, or a locally convex topological space. The purpose of this paper is to establish properties of Aumann's integrals of set-valued functions, $F: T \to 2^X$, whose values are nonempty subsets of a real separable reflexive Banach space X, and to continue the work due to Aumann [2] and Datko [7–8].

While previous analysis has always treated the case of special finite nonatomic measure spaces, we focus here on the case of general σ-finite nonatomic measure spaces. In this last situation, moreover, the analogous results we establish hold under less stringent hypotheses.

More precisely, all through the paper we consider a measure space (T, Σ, μ), where μ is supposed to be positive, nonatomic and σ-finite, and we give the following statements.

RESULT 1. *Let $F: T \to 2^X$ be a set-valued function. Then the closure of the Aumann integral of F, $\operatorname{cl} \int_T F(t)\, d\mu(t)$, is convex.*

This first theorem is a generalization of analogous results due to Richter [14] and to Aumann [2] in the finite-dimensional case.

RESULT 2 (REPRESENTATION THEOREM). *We assume that Σ possesses the Souslin operation and that $F: T \to 2^X$ is a set-valued function of Souslin type such that*

$$\int_T F(t)\, d\mu(t) \neq \varnothing.$$

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Then we have

(1) $\mathrm{cl} \int_T F(t)\, d\mu(t) = \{x \in X: x'(x) \leqslant \int_T s(x', F(t))\, d\mu(t), x' \in X'\}$;

(2) $s(x', \int_T F(t)\, d\mu(t)) = \int_T s(x', F(t))\, d\mu(t)$, $x' \in X'$;

(3) $\mathrm{cl} \int_T F(t)\, d\mu(t) = \mathrm{cl} \int_T \overline{\mathrm{co}}\, F(t)\, d\mu(t)$.

In 1974 Artstein established a representation theorem in the case that $X = \mathscr{R}^p$ and $T = [0, T]$ (cf. Lemma 2.2 of [1]). It should be noted that our version of the theorem includes Artstein's earlier result in this direction. Furthermore, the equality (3), which extends Theorem 3 of Aumann [2] to the infinite-dimensional case, is best possible even if $X = \mathscr{R}^p$, as we show in Remark (1) of Corollary 3.3.

RESULT 3 (LEBESGUE'S DOMINATED CONVERGENCE THEOREM). *Let Σ have the Souslin operation and let $(F_n)_n$ be a sequence of set-valued functions of Souslin type. We suppose that*

(α) *there exists $g \in L^1(T)$, $g \geqslant 0$, such that $\|F_n(t)\| \leqslant g(t)$, for $t \in T$, $n = 1, 2, \ldots$;*

(β) $\lim_n F_n(t) = F(t)$ *for $t \in T$, in terms of Definition* 2.4.

Then F is a set-valued function of Souslin type which maps T into nonempty closed bounded convex subsets of X and satisfies

$$\lim_n \int_T F_n(t)\, d\mu(t) = \int_T F(t)\, d\mu(t).$$

This theorem is an extension of Theorem 5 of Aumann [2] and makes use of a convergence which seems natural in the infinite-dimensional case (cf. Definition 2.4). More precisely, if $F_n(t)$ converges to $F(t)$ in the sense of Kuratowski, as imposed by Aumann in his Theorem 5, then $F_n(t)$ must converge to $F(t)$ in our topology. The converse to this statement is not true.

As Debreu observed in [9], from the viewpoint of economic interpretation, Aumann's assumption that the set of agents is an analytic set seems more artificial than his requirement that it is a measure space. For this reason, in [9] Debreu studies the integration of measurable set-valued functions whose values are nonempty compact convex subsets of a real Banach space.

According to Debreu's point of view, it seems of interest to establish the above Results 1, 2 and 3 for measurable closed-valued functions defined on (T, Σ, μ), where Σ is now supposed to be only μ-complete instead of having the Souslin operation. It is not difficult to show that the above results remain true in this new setting. A presentation of this fact will appear in the proofs of Theorem 3.2′ and its Corollary 3.3′, and in the note of Section 3. Also in this case, the representation theorem can be used to reveal some important properties of Aumann's integration, which themselves contain some well-known theorems in the literature (see, for instance, Datko [7, 8]).

2. Preliminaries

Let $X=(X, \|\cdot\|)$ be a real separable reflexive Banach space and $X'=(X', \|\cdot\|)$ the topological dual of X. Let S' denote the surface of the unit ball in X'.

Moreover, let (T, Σ, μ) be an arbitrary measure space, where Σ is a σ-algebra of subsets of T and μ is a positive, σ-finite, nonatomic measure. $L^1(T, X)$ denotes the usual Banach space of functions $\psi: T \to X$, where the norm is defined in the usual manner (see [10]). If $X=\mathscr{R}$, then $L^1(T, \mathscr{R})$ will be denoted by $L^1(T)$. Let 2^X be the family of all nonempty subsets of X.

Definition 2.1. A set-valued function $F: T \to 2^X$ is called *measurable* if the set $F^-(U)=\{t \in T: F(t) \cap U \neq \varnothing\}$ is measurable whenever $U \subset X$ is open. We define the *graph* of F, denoted $\mathrm{Gr}(F)$, by $\mathrm{Gr}(F)=\{(t,x) \in T \times X: x \in F(t)\}$. A *selector* of F is a function $\sigma: T \to X$ such that $\sigma(t) \in F(t)$, for $t \in T$. By $\mathscr{S}(F)$ we mean the family of all measurable selectors of F and by $\mathscr{L}(F)$ the set $\mathscr{S}(F) \cap L^1(T, X)$.

Definition 2.2. If a topological space P is separable and can be metrised so that it becomes a complete metric space, then P is said to be a *Polish space.* A set-valued function $F: T \to 2^X$ is said to be of *Souslin type* if there exists a Polish space P, a measurable set-valued function $\Omega: T \to 2^P$ with closed values and a continuous mapping $\phi: P \to X$ such that $F(t)=\phi(\Omega(t))$, for $t \in T$.

For a comprehensive survey of this topic, as well as other related topics, see [5, 11, 13, 15, 16].

Definition 2.3. Let A be a nonempty subset of X. We define $s(x', A)=\sup_{x \in A} x'(x)$, for $x' \in X'$. The function $s(\cdot, A): X' \to \mathscr{R}_\infty$, where $\mathscr{R}_\infty=\mathscr{R} \cup \{+\infty\}$, is said to be the *support function* of A.

Definition 2.4. Let $(A_n)_n$ be a given sequence in 2^X. We define its *lower limit* by

$$\varliminf_n A_n=\{x \in X: x'(x) \leqslant \varliminf_n s(x', A_n), x' \in X'\}$$

and its *upper limit* by

$$\varlimsup_n A_n=\{x \in X: x'(x) \leqslant \varlimsup_n s(x', A_n), x' \in X'\}.$$

The sequence $(A_n)_n$ is said to be *convergent* to A if $\underline{\lim}_n A_n = \overline{\lim}_n A_n = A$, and we put $\lim_n A_n = A$.

Note that $\underline{\lim}_n A_n$ and $\overline{\lim}_n A_n$ are two closed convex subsets of X and, furthermore, $\underline{\lim}_n A_n \subset \overline{\lim}_n A_n$.

Now, let $(A_n)_n$ be a fixed equibounded sequence in 2^X; in other words, there exists $M > 0$ such that $M \geqslant \|A_n\| = \sup_{x \in A_n} \|x\|$, for $n = 1, 2, \dots$.

Remarks. (1) Since $(A_n)_n$ is equibounded, it is clear that $p(x') = \overline{\lim}_n s(x', A_n)$, $x' \in X'$, is a continuous sublinear functional. Thus, from Theorem II-16 of [5] it follows that p is the support function of the nonempty closed bounded convex set $\overline{\lim}_n A_n$. Therefore, we get

$$s(x', \overline{\lim_n} A_n) = \overline{\lim_n}\, s(x', A_n), \qquad x' \in X',$$

and, if $\underline{\lim}_n A_n \neq \varnothing$, then

$$s(x', \underline{\lim_n} A_n) \leqslant \underline{\lim_n}\, s(x', A_n), \qquad x' \in X'.$$

(2) We now prove that $\overline{\lim}_n A_n = \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{m=n}^{\infty} A_m$. The inclusion $\overline{\lim}_n A_n \subset \bigcap_{n=1}^{\infty} \overline{\text{co}} \bigcup_{m=n}^{\infty} A_m$ is clear. To show the converse inclusion it is sufficient to note that for $n = 1, 2, \dots$ we have

$$\sup_{m \geqslant n} s(x', A_m) = s\left(x', \overline{\text{co}} \bigcup_{m=n}^{\infty} A_m\right), \qquad x' \in X';$$

in other words the only assertion which is not immediately evident is that

$$s\left(x', \overline{\text{co}} \bigcup_{m=n}^{\infty} A_m\right) \leqslant \sup_{m \geqslant n} s(x', A_m), \qquad x' \in X'.$$

In order to see this last inequality, we assume, by contradiction, that there exist n and x' such that $s(x', \overline{\text{co}} \bigcup_{m=n}^{\infty} A_m) > \sup_{m \geqslant n} s(x', A_m)$. Then we can choose $x_0 \in \text{co} \bigcup_{m=n}^{\infty} A_m$, with $x_0 = \sum_{i=1}^{q} \lambda_i y_{m_i}$, $\lambda_i \geqslant 0$, $\sum_{i=1}^{q} \lambda_i = 1$, $y_{m_i} \in A_{m_i}$, $m_i \geqslant n$ $(i = 1, \dots, q)$, such that $x'(x_0) > \sup_{m \geqslant n} s(x', A_m)$. Finally, it follows that $x'(x_0) = \sum_{i=1}^{q} \lambda_i x'(y_{m_i}) \leqslant \sum_{i=1}^{q} \lambda_i s(x', A_{m_i}) \leqslant \sup_{m \geqslant n} s(x', A_m) < x'(x_0)$, which is absurd.

(3) Let $v = \{x'_i\}_{i=1}^{\infty}$ be a countable dense subset of S'. We shall prove that

$$\underline{\lim_n} A_n = \bigcap_{i=1}^{\infty} \{x \in X : x'_i(x) \leqslant \underline{\lim_n}\, s(x'_i, A_n)\},$$

$$\overline{\lim_n} A_n = \bigcap_{i=1}^{\infty} \{x \in X : x'_i(x) \leqslant \overline{\lim_n}\, s(x'_i, A_n)\}.$$

We begin by showing the first equality.

From Definition 2.4 it follows that $\underline{\lim}_n A_n \subset \bigcap_{i=1}^{\infty} \{x \in X\colon\ x_i'(x) \leqslant \underline{\lim}_n s(x_i', A_n)\}$. Now let $x \in X$ be such that $x_i'(x) \leqslant \underline{\lim}_n s(x_i', A_n)$, for $i = 1, 2, \ldots$. Let $x' \in X'$ and $\varepsilon > 0$ be fixed. We may assume $\|x'\| > 0$; otherwise there is nothing to prove. We now consider $x_i' \in v$ such that $\|x'/\|x'\| - x_i'\| < \varepsilon/\|x'\|\,(M + \|x\|)$. Since we have

$$|s(\|x'\|\, x_i', A_n) - s(x', A_n)| \leqslant \|\, \|x'\|\, x_i' - x'\| \cdot \|A_n\| < \varepsilon M/(M + \|x\|), \qquad n = 1, 2, \ldots,$$

we can derive that

$$\|x'\| \,\underline{\lim}_n\, s(x_i', A_n) \leqslant \underline{\lim}_n\, s(x', A_n) + \varepsilon M/(M + \|x\|).$$

Therefore, we get

$$x'(x) \leqslant \|x'\|\, x_i'(x) + \varepsilon\, \|x\|/(M + \|x\|) \leqslant \|x'\| \,\underline{\lim}_n\, s(x_i', A_n) + \varepsilon\, \|x\|/(M + \|x\|) \leqslant \underline{\lim}_n\, s(x', A_n) + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $x'(x) \leqslant \underline{\lim}_n s(x', A_n)$; this holds for all $x' \in X'$, so $x \in \underline{\lim}_n A_n$. Similarly, we can prove the second equality.

(4) Let v be as in Remark (3). The family of nonempty closed bounded convex subsets of X will be denoted by $\overline{\mathrm{co}}\, \mathscr{K}(X)$. Following Definition 5 of [7] we consider the distance function

$$d_v(A, B) = \sum_{i=1}^{\infty} \frac{1}{2^i} \frac{|s(x_i', A) - s(x_i', B)|}{1 + |s(x_i', A) - s(x_i', B)|},$$

defined on $\overline{\mathrm{co}}\, \mathscr{K}(X) \times \overline{\mathrm{co}}\, \mathscr{K}(X)$.

We now assume that the fixed sequence $(A_n)_n$ is in $\overline{\mathrm{co}}\, \mathscr{K}(X)$. As Datko observed in Remark 2 of [7], one can prove that the sequence $(A_n)_n$ converges in terms of the metric d_v to a set A of $\overline{\mathrm{co}}\, \mathscr{K}(X)$ if and only if

$$\lim_n s(x_i', A_n) = s(x_i', A), \qquad \text{for} \quad i = 1, 2, \ldots.$$

On account of Remark (1) it follows that $\lim_n A_n = A$ if and only if

$$\lim_n s(x', A_n) = s(x', A), \qquad \text{for} \quad x' \in X'.$$

Therefore, by Remark (3), $\lim_n s(x', A_n) = s(x', A)$, for $x' \in X'$, if and only if $\lim_n s(x_i', A_n) = s(x_i', A)$, $i = 1, 2, \ldots$, for every countable dense subset $v = \{x_i'\}_{i=1}^{\infty}$ of S', in other words, $\lim_n A_n = A$ if and only if $\lim_n d_v(A_n, A) = 0$, for every v.

DEFINITION 2.5. The σ-algebra Σ admits the *Souslin operation* $\mathscr{A}$ if $\mathscr{A}(\Sigma)=\Sigma$. Further details on this topic can be found in [12, 13].

DEFINITION 2.6. A sequence $(F_n)_n$, $F_n: T\to 2^X$, is said to be *integrably bounded* if there exists $g\in L^1(T)$, $g\geqslant 0$, such that $\|F_n(t)\|\leqslant g(t)$, for $t\in T$.

DEFINITION 2.7. Let $F: T\to 2^X$ be a set-valued function. The *Aumann integral* of F is defined as

$$\int_T F(t)\,d\mu(t)=\left\{\int_T \sigma(t)\,d\mu(t): \sigma\in\mathscr{L}(F)\right\}.$$

Instead of $\int_T F(t)\,d\mu(t)$, $\int_T \sigma(t)\,d\mu(t)$, etc., we shall write $\int F(t)\,d\mu(t)$, $\int \sigma(t)\,d\mu(t)$, etc.

No confusion will arise if we mean "for almost every" when we write "for" $t\in T$.

3. THE REPRESENTATION THEOREM AND SOME APPLICATIONS

We begin the present section with the following

THEOREM 3.1. *Let $F: T\to 2^X$ a set-valued function. Then,* cl $\int F(t)\,d\mu(t)$ *is a convex subset of X.*

Proof. Without loss of generality we can assume that cl $\int F(t)\,d\mu(t)$ contains at least two points. We first prove that if $r_1, r_2\in\int F(t)\,d\mu(t)$ then, for every $\varepsilon>0$ and every a with $0<a<1$, there exists a point $r\in\int F(t)\,d\mu(t)$ such that $\|r-ar_1-(1-a)\,r_2\|<\varepsilon$.

We now fix r_1, r_2, ε and a as above. Therefore, there exist $\sigma_1, \sigma_2\in\mathscr{L}(F)$ such that $r_1=\int\sigma_1(t)\,d\mu(t)$, $r_2=\int\sigma_2(t)\,d\mu(t)$. We denote by $(\phi_{1n})_n$ and $(\phi_{2n})_n$ two sequences of integrable step functions such that $\sup_{R\in\Sigma}\int_R\|\sigma_i(t)-\phi_{in}(t)\|\,d\mu(t)<\varepsilon/3$, for every $n\geqslant N=N(\varepsilon/3)$, $i=1, 2$. Thus, it is no restriction of generality to write $\phi_{in}=\sum_{j=1}^{M}\chi_{S_j}x_{i,j}$, $i=1,2$, where $S_j\in\Sigma$, $S_j\cap S_h=\varnothing$, $j\neq h$, $j, h=.1, 2,..., M$.

By the corollary of [10, p. 28], μ has the Darboux property; in other words, there exist $R_1,..., R_M\in\Sigma$, $R_j\subset S_j$, $\mu(R_j)=a\mu(S_j)$, $R_j\cap R_h=\varnothing, j\neq h$, $j, h=1, 2,..., M$. Put $E=\bigcup_{j=1}^{M}R_j$, $E^c=T-E$; thus we have

$$a\int\phi_{1n}(t)\,d\mu(t)=\int_E\phi_{1n}(t)\,d\mu(t),$$

$$(1-a)\int\phi_{2n}(t)\,d\mu(t)=\int_{E^c}\phi_{2n}(t)\,d\mu(t).$$

Furthermore, if $\sigma = \chi_E \sigma_1 + \chi_{E^c} \sigma_2$, then $\sigma \in \mathscr{L}(F)$ and $r = \int \sigma(t)\, d\mu(t) \in \int F(t)\, d\mu(t)$. Finally, we get

$$\begin{aligned}
\|r - ar_1 - (1-a)\, r_2\| &\leqslant \left\| \int_E \sigma(t)\, d\mu(t) - \int_E \phi_{1n}(t)\, d\mu(t) \right.\\
&\quad \left. + \int_E \phi_{1n}(t)\, d\mu(t) - a \int \sigma_1(t)\, d\mu(t) \right\|\\
&\quad + \left\| \int_{E^c} \sigma(t)\, d\mu(t) - \int_{E^c} \phi_{2n}(t)\, d\mu(t) \right.\\
&\quad \left. + \int_{E^c} \phi_{2n}(t)\, d\mu(t) - (1-a) \int \sigma_2(t)\, d\mu(t) \right\|\\
&\leqslant \int_E \|\sigma_1(t) - \phi_{1n}(t)\|\, d\mu(t) + a \int \|\phi_{1n}(t) - \sigma_1(t)\|\, d\mu(t)\\
&\quad + \int_{E^c} \|\sigma_2(t) - \phi_{2n}(t)\|\, d\mu(t) + (1-a)\\
&\qquad \int \|\phi_{2n}(t) - \sigma_2(t)(t)\|\, d\mu(t)\\
&< \varepsilon/3 + a\varepsilon/3 + \varepsilon/3 + (1-a)\, \varepsilon/3 = \varepsilon,
\end{aligned}$$

which is the desired conclusion.

We are now able to prove that $\mathrm{cl} \int F(t)\, d\mu(t)$ is convex. For this purpose, let us fix $r_1, r_2 \in \mathrm{cl} \int F(t)\, d\mu(t)$ and $0 < a < 1$. Corresponding to each $n = 1, 2, \ldots$, there exist $r_{1n}, r_{2n} \in \int F(t)\, d\mu(t)$ such that $\|r_i - r_{in}\| < 1/2n$, $i = 1, 2$. As we showed above, we can choose $\bar{r}_n \in \int F(t)\, d\mu(t)$ such that

$$\|\bar{r}_n - ar_{1n} - (1-a) r_{2n}\| < 1/2n.$$

Finally, we have, for $n = 1, 2, \ldots$,

$$\begin{aligned}
\|\bar{r}_n - ar_1 - (1-a)\, r_2\| &\leqslant \|\bar{r}_n - ar_{1n} - (1-a)\, r_{2n}\| + a\, \|r_{1n} - r_1\|\\
&\quad + (1-a)\, \|r_{2n} - r_2\| < 1/n;
\end{aligned}$$

in other words, the sequence $(\bar{r}_n)_n$ in $\int F(t)\, d\mu(t)$ converges to $ar_1 + (1-a)\, r_2$ and thus the theorem is proved.

Remark. This theorem was established in Euclidean spaces by Richter [14] and, subsequently, by Aumann [2]. Furthermore, although the proof we present here is completely analogous to one due to Datko in [6], our statement contains Theorem 1 of [6] as well as the last theorem in Section 5 of [7]. We point out that the measure we consider is possibly infinite.

The next theorem will be of importance for the coming applications.

THEOREM 3.2 (REPRESENTATION THEOREM). *We suppose that Σ has the Souslin operation. Let $F: T \to 2^X$ be a set-valued function of Souslin type such that $\int F(t)\, d\mu(t) \neq \varnothing$. Then we have*

$$\mathrm{cl} \int F(t)\, d\mu(t) = \left\{ x \in X : x'(x) \leqslant \int s(x', F(t))\, d\mu(t),\ x' \in X' \right\},$$

$$s\left(x', \int F(t)\, d\mu(t)\right) = \int s(x', F(t))\, d\mu(t).$$

Proof. For each $x' \in X'$ we set $s(x', t) = s(x', F(t))$, for $t \in T$. In view of Theorem 7 in [13], the function $s(x', \cdot): T \to \mathscr{R}_\infty$ is measurable, since it is the supremum of a sequence of measurable functions.

Obviously, there exists some $\sigma \in \mathscr{L}(F)$ and the function $g(t) = \|\sigma(t)\|$, for $t \in T$, is such that $g \in L^1(T)$, $g \geqslant 0$ and

$$-\|x'\|\, g(t) \leqslant s(x', t), \qquad \text{for all } x' \in X'. \tag{+}$$

Hence, $s(x', \cdot)$ is integrable, with finite or $+\infty$ integral. Thus, we can define $s: X' \to \mathscr{R}_\infty$ by

$$s(x') = \int s(x', t)\, d\mu(t), \qquad x' \in X'.$$

It is clear that s is a sublinear functional. We now prove that s is lower semicontinuous.

For this purpose, we fix $x' \in X'$ and a sequence $(x'_n)_n$ in X' such that $\lim_n x'_n = x'$. By the lower semicontinuity of support functions it follows that $s(x', t) \leqslant \underline{\lim}_n\, s(x'_n, t)$, for $t \in T$. Furthermore, since $\|x'_n\| \leqslant M$, $n = 1, 2, \ldots$, for some $M > 0$, from (+) we have $-Mg(t) \leqslant s(x'_n, t)$, for $n = 1, 2, \ldots$ and $t \in T$. Hence, in view of Fatou's Lemma, the desired inequality

$$s(x') \leqslant \int \underline{\lim}_n\, s(x'_n, t)\, d\mu(t) \leqslant \underline{\lim}_n \int s(x'_n, t)\, d\mu(t) = \underline{\lim}_n\, s(x'_n)$$

holds.

Note that $s(0) = 0$. Therefore, on account of Theorem II-16 in [5] and the reflexivity of X, there exists a unique subset H of X such that H is nonempty, closed and convex and $s(x') = s(x', H)$ holds for each $x' \in X'$. In addition, $H = \{x \in X : x'(x) \leqslant s(x'), x' \in X'\}$.

We now show that $s(x') = s(x', \int F(t)\, d\mu(t))$ for every $x' \in X'$. We first observe that, if $r = \int \sigma(t)\, d\mu(t)$, where $\sigma \in \mathscr{L}(F)$, then for all $x' \in X'$ it follows that $x'(r)=\int x'(\sigma(t))\,d\mu(t)\leqslant\int s(x',t)\,d\mu(t)=s(x')$; in other words, $s(x',\int F(t)\,d\mu(t))\leqslant s(x')$. It remains to establish the reverse inequality. Let $(T_n)_n$ be a fixed sequence in Σ such that $0<\mu(T_n)<+\infty$, $T_n\cap T_m=\varnothing$, $n\neq m$, $n,m=1,2,\ldots$, $\bigcup_{n=1}^{\infty}T_n=T$. We consider the sequence $(f_m)_m$ of functions $f_m: T\to\mathscr{R}^+$ defined by $f_m(t)=1/m2^n\mu(T_n)$, $t\in T_n$, $n=1,2,\ldots$, where $\int f_m(t)\,d\mu(t)=1/m$, $m=1,2,\ldots$. We now choose $x'\in X'$. Set $R=\{t\in T: s(x',t)<+\infty\}$ and $S=T-R$; we define the sequence of measurable functions $\phi_m: T\to\mathscr{R}$, $\phi_m(t)=s(x',t)-f_m(t)$, $t\in R$, $\phi_m(t)=m$, $t\in S$, $m=1,2,\ldots$. For each $m=1,2,\ldots$, by construction there holds

$$-g(t)\,\|x'\|-f_m(t)\leqslant\phi_m(t)<s(x',t),\qquad\text{for}\quad t\in T,$$

$$\int\phi_m(t)\,d\mu(t)=\int_R[s(x',t)-f_m(t)]\,d\mu(t)+m\mu(S).$$

Thus we have

$$\lim_m\int\phi_m(t)\,d\mu(t)=\begin{cases}s(x'), & \text{if}\quad\mu(S)=0,\\ +\infty, & \text{if}\quad\mu(S)>0;\end{cases}$$

in other words, in both cases $\lim_m\int\phi_m(t)\,d\mu(t)=s(x')$.

Let m be a fixed integer. Put $G_m(t)=\{x\in F(t): x'(x)\geqslant\phi_m(t)\}$; we note that $G_m(t)\neq\varnothing$, since $\phi_m(t)<s(x',t)$, for $t\in T$. Set $H_m(t)=\{x\in X: x'(x)\geqslant\phi_m(t)\}$, for $t\in T$; it is evident that $\mathrm{Gr}(H_m)\in\Sigma\otimes\mathscr{B}(X)$. Furthermore, on account of Corollary 5.4 in [12], $\mathrm{Gr}(F)$ belongs to $\mathscr{A}(\Sigma\otimes\mathscr{B}(X))$. From the equality $\mathrm{Gr}(G_m)=\mathrm{Gr}(F)\cap\mathrm{Gr}(H_m)$, we deduce that $\mathrm{Gr}(G_m)\in\mathscr{A}(\Sigma\otimes\mathscr{B}(X))$, and by the same corollary it follows that G_m is of Souslin type.

In view of Theorem 7 in [13], there exists a measurable selector $\psi: T\to X$ of G_m. For every $q=1,2,\ldots$, we consider the measurable sets $R_{n,q}=\{t\in T_n: \|\psi(t)\|\leqslant q/2^n\mu(T_n)\}$ and $S_{n,q}=T_n-R_{n,q}$, $n=1,2,\ldots$. Let $R_q=\bigcup_{n=1}^{\infty}R_{n,q}$ and $S_q=\bigcup_{n=1}^{\infty}S_{n,q}$.

It is clear that $T=R_q\cup S_q$ and $R_q\cap S_q=\varnothing$, $T=\bigcup_q R_q$, $R_q\subset R_{q+1}$, $\varnothing=\bigcap_q S_q$, $S_q\supset S_{q+1}$, $q=1,2,\ldots$. On account of requirements, we can choose $\sigma\in\mathscr{L}(F)$. Finally, we define $\eta_q(t)=\psi(t)$, $t\in R_{n,q}$, $\eta_q(t)=\sigma(t)$, $t\in S_{n,q}$, $n,q=1,2,\ldots$, and again fix $q=1,2,\ldots$. Because $\eta_q\in\mathscr{S}(F)$ and

$$\begin{aligned}\int\|\eta_q(t)\|\,d\mu(t)&=\sum_{n=1}^{\infty}\int_{R_{n,q}}\|\psi(t)\|\,d\mu(t)+\sum_{n=1}^{\infty}\int_{S_{n,q}}\|\sigma(t)\|\,d\mu(t)\\&\leqslant\sum_{n=1}^{\infty}\int_{T_n}q\,d\mu(t)/2^n\mu(T_n)+\sum_{n=1}^{\infty}\int_{T_n}\|\sigma(t)\|\,d\mu(t)\\&=q+\int\|\sigma(t)\|\,d\mu(t),\end{aligned}$$

we get $n_q\in\mathscr{L}(F)$; in other words, $r_q=\int\eta_q(t)\,d\mu(t)\in\int F(t)\,d\mu(t)$.

In addition, we have

$$s\left(x', \int F(t)\, d\mu(t)\right) \geqslant x'(r_q) = \sum_{n=1}^{\infty} \int_{T_n} x'(\eta_q(t))\, d\mu(t)$$

$$= \sum_{n=1}^{\infty} \int_{R_{n,q}} x'(\psi(t))\, d\mu(t) + \sum_{n=1}^{\infty} \int_{S_{n,q}} x'(\sigma(t))\, d\mu(t)$$

$$= \int_{R_q} x'(\psi(t))\, d\mu(t) + \int_{S_q} x'(\sigma(t))\, d\mu(t)$$

$$\geqslant \int_{R_q} \phi_m(t)\, d\mu(t) + \int_{S_q} x'(\sigma(t))\, d\mu(t),$$

where the last inequality is true since $\psi \in \mathscr{S}(G_m)$. Since $q = 1, 2, \ldots$ is arbitrary, we obtain

$$s\left(x', \int F(t)\, d\mu(t)\right) \geqslant \lim_q \left[\int_{R_q} \phi_m(t)\, d\mu(t) + \int_{S_q} x'(\sigma(t))\, d\mu(t)\right]$$

$$= \int \phi_m(t)\, d\mu(t).$$

Observing that this inequality holds for each $m = 1, 2, \ldots$, we conclude that

$$s\left(x', \int F(t)\, d\mu(t)\right) \geqslant s(x').$$

It is now evident, from what has been proved above, that

$$s\left(x', \int F(t)\, d\mu(t)\right) = s(x') = \int s(x', F(t))\, d\mu(t), \qquad \text{for all } x' \in X'.$$

In view of Theorem 3.1 and the Hahn–Banach Theorem it follows finally that $\operatorname{cl} \int_T F(t)\, d\mu(t) = H$.

THEOREM 3.2′. *We assume that Σ is μ-complete. Let F be a measurable set-valued function from T into nonempty closed subsets of X such that $\int F(t)\, d\mu(t) \neq \phi$. Then the conclusion of Theorem* 3.2 *is still true.*

Proof. On account of Example (i) and Theorem 7 in [13], the set-valued function F has a Castaing representation. Under our requirements, this property is equivalent to the measurability of F as well as to the fact that $\operatorname{Gr}(F) \in \Sigma \otimes \mathscr{B}(X)$, in view of Theorem III-30 in [5].

Following the notation we used in the proof of Theorem 3.2, we observe that in the present case the set-valued function G_m maps T into nonempty closed subsets of X. Replacing Corollary 5.4 of [12] and Theorem 7 of [13] by Theorem III-30 of [5], we obtain the measurability of G_m and the existence of a measurable selector ψ of G_m. The remaining part of the proof is the same as for Theorem 3.2.

Remark. Theorem 3.2 is inspired by the representation theorem of Artstein [1, Lemma 2.2] in the case that $X = \mathscr{R}^p$. However, Artstein's result is a special case of our representation theorem.

COROLLARY 3.3. *Under the assumptions of Theorem* 3.2, *we have the equality*

$$\mathrm{cl} \int F(t)\, d\mu(t) = \mathrm{cl} \int \overline{\mathrm{co}}\, F(t)\, d\mu(t).$$

Proof. From the corollary of Theorem 4 in [13] we see that $(\overline{\mathrm{co}}\, F)(t) = \overline{\mathrm{co}}\,(F(t))$, $t \in T$, is a set-valued function of Souslin type and, thus, by the representation Theorem 3.2, it is easily seen that

$$\begin{aligned} \mathrm{cl} \int F(t)\, d\mu(t) &= \left\{ x \in X \colon x'(x) \leqslant \int s(x', F(t))\, d\mu(t),\, x' \in X' \right\} \\ &= \left\{ x \in X \colon x'(x) \leqslant \int s(x', \overline{\mathrm{co}}\, F(t))\, d\mu(t),\, x' \in X' \right\} \\ &= \mathrm{cl} \int \overline{\mathrm{co}}\, F(t)\, d\mu(t). \end{aligned}$$

COROLLARY 3.3′. *The assumptions of Theorem* 3.2′ *imply that*

$$\mathrm{cl} \int F(t) d\mu(t) = \mathrm{cl} \int \overline{\mathrm{co}}\, F(t)\, d\mu(t).$$

Proof. In view of Theorem III-40 in [5], $\overline{\mathrm{co}}\, F$ is measurable and the equality follows from Theorem 3.2′, as shown in the proof of the previous corollary.

Remark (1). Note that the statements of Corollaries 3.3 and 3.3′ are best possible. Indeed, even in the finite-dimensional case, $\int \overline{\mathrm{co}}\, F(t)\, d\mu(t)$ need not be closed and, furthermore, it is evident that in general $\int F(t)\, d\mu(t) \neq \int \mathrm{co}\, F(t) d\mu(t)$. We next give a slight modification of an example due to Aumann [2] which can be employed to show the first fact.

Let $T =]0, 1[$, Σ be the Lebesgue σ-algebra and μ the Lebesgue measure. Define $F \colon T \to 2^{\mathscr{R}^2}$ by setting $F(t) = \{(0, 0), ((1-t)/t, t/(1-t))\}$, $t \in T$. Obviously, $F(t) = \overline{F(t)}$, $t \in T$, and F has a Castaing representation. Thus, F

is measurable as well as of Souslin type, by the same arguments used in the previous proofs. Furthermore, the hypotheses of Theorem 3.2 are fulfilled. We now observe that $\sigma: T \to \mathscr{R}^2$ is a measurable selector of $\overline{\text{co}}\, F$ if and only if

$$\sigma(t) = \left(\lambda(t)\frac{1-t}{t}, \lambda(t)\frac{t}{1-t}\right),$$

$t \in T$, where $\lambda: T \to [0, 1]$ is measurable.

In addition, if $(x, y) \in \int \overline{\text{co}}\, F(t)\, d\mu(t)$ and if $x = 0$ $(y = 0)$, then $y = 0$ $(x = 0)$. For each $n = 1, 2,\ldots$, we consider $\lambda_n = \chi_{[1/4n, 1/2n]}$ and let σ_n be the corresponding measurable selector of $\overline{\text{co}}\, F$. Thus, we obtain

$$\lim_n \int \sigma_n(t)\, d\mu(t) = \lim_n \left(\log 2 - 1/4n, \log \frac{4n-1}{4n-2} - 1/4n\right) = (\log 2, 0).$$

Hence $(\log 2, 0) \notin \int \overline{\text{co}}\, F(t)\, d\mu(t)$, and this completes the example.

Remark (2). Corollary 3.3 extends an analogous result of Aumann [2, Theorem 3] to the infinite-dimensional case.

Note. In the remaining part of this section we consider only set-valued functions of Souslin type which are defined on (T, Σ, μ), and we assume that $\mathscr{A}(\Sigma) = \Sigma$. Obviously, by the same arguments used in the proofs of Theorem 3.2′ and Corollary 3.3′, all the next results can also be established for measurable closed-valued functions defined on (T, Σ, μ). We stress the fact, however, that the requirement that $\mathscr{A}(\Sigma) = \Sigma$ is always replaced by the assumption that Σ is μ-complete.

THEOREM 3.4. *Let $F: T \to 2^X$ be an integrably bounded set-valued function of Souslin type. Then the set $\int \overline{\text{co}}\, F(t)\, d\mu(t)$ is closed.*

Proof. Let $r \in \text{cl} \int \overline{\text{co}}\, F(t)\, d\mu(t)$ and $(\sigma_n)_n$ be a sequence in $\mathscr{L}(\overline{\text{co}}\, F)$ such that $r = \lim_n \int \sigma_n(t)\, d\mu(t)$. We first observe that Definition 2.6 implies $\|\sigma_n(t)\| \leqslant g(t)$, for $t \in T$, $n = 1, 2,\ldots$. Then, $K = \{\sigma_n\}_{n=1}^{\infty} \subset L^1(T, X)$ satisfies the hypotheses of Theorem 1, part I-b in [4]; in other words, K is relatively weakly compact in $L^1(T, X)$. Therefore, without loss of generality we may assume that the whole original sequence $(\sigma_n)_n$ converges weakly to some $\sigma \in L^1(T, X)$. In view of the Mazur Theorem there exists a suitable sequence $(\psi_n)_n$ of convex combinations of $(\sigma_n)_n$ which converges strongly to σ in $L^1(T, X)$. It is now evident that, by choosing subsequences if necessary, we may claim that $\lim_n \psi_n(t) = \sigma(t)$, for $t \in T$, hence $\sigma(t) \in \overline{\text{co}}\, F(t)$, for $t \in T$. Since the operator $\int: L^1(T, X) \to X$ is strongly continuous and, therefore, weakly continuous, it follows that $r = \lim_n \int \sigma_n(t)\, d\mu(t) = \int \sigma(t)\, d\mu(t)$. The last equality concludes the proof.

Remark. Under the assumptions of Theorem 3.4, it is also immediate from Corollary 3.3 that $\mathrm{cl} \int F(t)\, d\mu(t) = \int \overline{\mathrm{co}}\, F(t)\, d\mu(t)$. Thus on account of the previous note it follows that this statement contains the main theorem of [8] as a special case. It is to be noted that the measure we consider is possibly infinite.

THEOREM 3.5. *Let* $(F_n)_n$, $F_n: T \to 2^X$, $n = 1, 2, \ldots$, *be an integrably bounded sequence of set-valued functions of Souslin type. Then, the following inclusion hold*:

(a) $\int \underline{\lim}_n F_n(t)\, d\mu(t) \subset \underline{\lim}_n \int F_n(t)\, d\mu(t)$,

(b) $\int \overline{\lim}_n F_n(t)\, d\mu(t) \supset \overline{\lim}_n \int F_n(t)\, d\mu(t)$.

Proof. (a) By setting $F(t) = \underline{\lim}_n F_n(t)$, for $t \in T$, we obtain from Definition 2.4 and Remark (2) of Section 2 that $F(t) = \{x \in X: x'(x) \leqslant \underline{\lim}_n s(x', F_n(t)), x' \in X'\} = \bigcap_{i=1}^{\infty} \{x \in X: x'_i(x) \leqslant \underline{\lim}_n s(x'_i, F_n(t))\}$, for $t \in T$, where $\{x'_i\}_{i=1}^{\infty}$ is a fixed countable dense subset of S'.

Therefore, $F(t)$ is a closed bounded convex subset of X, for $t \in T$. We may assume that $F(t) \neq \varnothing$, for $t \in T$; otherwise there is nothing to prove. Since F is a countable intersection of set-valued functions of Souslin type, then on account of Corollary 2 of Section 3 in [13] we claim that also F is of the same type. In view of Fatou's Lemma we have

$$\int \underline{\lim}_n s(x', F_n(t))\, d\mu(t) \leqslant \underline{\lim}_n \int s(x', F_n(t))\, d\mu(t),$$

and, hence, by Theorems 3.4 and 3.2 and Definition 2.4 it follows that

$$\begin{aligned} \int F(t)\, d\mu(t) = \mathrm{cl} \int F(t)\, d\mu(t) &= \left\{ x \in X: x'(x) \leqslant \int s(x', F(t))\, d\mu(t), x' \in X' \right\} \\ &\subset \left\{ x \in X: x'(x) \leqslant \int \underline{\lim}_n s(x', F_n(t))\, d\mu(t), x' \in X' \right\} \\ &\subset \left\{ x \in X: x'(x) \leqslant \underline{\lim}_n \int s(x' F_n(t))\, d\mu(t), x' \in X' \right\} \\ &= \underline{\lim}_n \int F_n(t)\, d\mu(t). \end{aligned}$$

(b) Letting $G(t) = \overline{\lim}_n F_n(t)$, from Definition 2.4 and Remark (3) of Section 2 we see that $G(t) = \{x \in X: x'(x) \leqslant \overline{\lim}_n s(x', F_n(t)), x' \in X'\} = \bigcap_{n=1}^{\infty} \overline{\mathrm{co}} \bigcup_{m=n}^{\infty} F_m(t)$; in other words, $G(t)$ is a nonempty closed bounded convex subset of X, for $t \in T$. On account of Theorem 1, corollary of Theorem 4 and Corollary 2 of Theorem 2 in [13], we may claim that G is of Souslin type.

Using Theorems 3.4 and 3.2, Fatou's Lemma and Definition 2.4, we get

$$\int G(t)\,d\mu(t) = \mathrm{cl}\int G(t)\,d\mu(t) = \left\{x \in X : x'(x) \leqslant \int s(x', G(t))\,d\mu(t), x' \in X'\right\}$$

$$= \left\{x \in X : x'(x) \leqslant \int \overline{\lim_n}\, s(x', F_n(t))\,d\mu(t), x' \in X'\right\}$$

$$\supset \left\{x \in X : x'(x) \leqslant \overline{\lim_n} \int s(x', F_n(t))\,d\mu(t), x' \in X'\right\}$$

$$= \overline{\lim_n} \int F_n(t)\,d\mu(t).$$

Thus the theorem is proved.

COROLLARY 3.6 (LEBESGUE'S DOMINATED CONVERGENCE THEOREM). *Let $(F_n)_n$ be an integrably bounded sequence of set-valued functions of Souslin type. If* $\lim_n F_n(t) = F(t)$, *for $t \in T$, then F is a set-valued function of Souslin type which maps T into nonempty closed bounded convex subsets of X and is such that*

$$\lim_n \int F_n(t)\,d\mu(t) = \int F(t)\,d\mu(t).$$

Remark. Corollary 3.6 is an extension of Theorem 5 in [2] and makes use of a convergence which seems rather natural in the infinite-dimensional case. In particular, when $X = \mathscr{R}^p$, Corollary 3.6 contains Theorem 5 of [2] as a special case. In order to see this, we denote by $\mathrm{Lim}\, A_n$ the Kuratowski limit of a sequence $(A_n)_n$, where each A_n is a subset of $\mathscr{R}^p$, and by h the Hausdorff metric in $\overline{\mathrm{co}}\, \mathscr{K}(\mathscr{R}^p)$.

We next prove that if $(F_n)_n$, $F_n : T \to 2^{\mathscr{R}^p}$, is a sequence of set-valued functions satisfying the assumptions of Theorem 5 in [2], then $(F_n)_n$ verifies the hypotheses of Corollary 3.6. In addition, the statement of Corollary 3.6 implies the conclusion of Theorem 5 by Aumann.

It is evident that in order to see that $(F_n)_n$ fulfills the requirements of Corollary 3.6 we need only show that $\mathrm{Lim}\, F_n(t) = F(t)$, for $t \in T$, implies $\lim_n F_n(t) = F(t)$, for $t \in T$. Let $t \in T$ be fixed. On account of Lemma 1.6 in [3] it follows that if $\mathrm{Lim}\, F_n(t) = F(t)$, then $\mathrm{Lim}\, \overline{\mathrm{co}}\, F_n(t) = \overline{\mathrm{co}}\, F(t)$. We stress the fact, however, that the converse of this implication need not be true. Now, by Corollary 1.2 and Lemma 1.4 of [3] we have $\mathrm{Lim}\, \overline{\mathrm{co}}\, F_n(t) = \overline{\mathrm{co}}\, F(t)$ if and only if $\lim_n h(\overline{\mathrm{co}}\, F_n(t), \overline{\mathrm{co}}\, F(t)) = 0$. From Remark (1) of Section 2 we see that $\lim_n s(x', \overline{\mathrm{co}}\, F_n(t)) = s(x', \overline{\mathrm{co}}\, F(t))$, $x' \in \mathscr{R}^p$, if and only if $\lim_n \overline{\mathrm{co}}\, F_n(t) = \overline{\mathrm{co}}\, F(t)$. In view of Lemma 1.2(iii) in [3] we have $\lim_n h(\overline{\mathrm{co}}\, F_n(t), \overline{\mathrm{co}}\, F(t)) = 0$ if and only if $\lim_n \overline{\mathrm{co}}\, F_n(t) = \overline{\mathrm{co}}\, F(t)$. Thus, Corollary 3.6 allows us to claim that $\lim_n \int \overline{\mathrm{co}}\, F_n(t)\,d\mu(t) = \int \overline{\mathrm{co}}\, F(t)\,d\mu(t)$.

By the arguments used above, we are now able to show that the last equality is equivalent to the assertion of Theorem 5 in [2]. In fact, the following implications hold:

$$\lim_n \int \overline{\text{co}}\, F_n(t)\, d\mu(t) = \int \overline{\text{co}}\, F(t)\, d\mu(t)$$

$$\Leftrightarrow \lim_n h\left(\int \overline{\text{co}}\, F_n(t)\, d\mu(t), \int \overline{\text{co}}\, F(t)\, d\mu(t)\right) = 0$$

$$\Leftrightarrow \text{Lim} \int \overline{\text{co}}\, F_n(t)\, d\mu(t) = \int \overline{\text{co}}\, F(t)\, d\mu(t)$$

$$\Leftrightarrow \text{Lim cl} \int F_n(t)\, d\mu(t) = \text{cl} \int F(t)\, d\mu(t) = \int F(t)\, d\mu(t),$$

where the last equality is a consequence of the fact that $F(t) = \text{Lim}\, F_n(t)$, for $t \in T$, is a closed-valued function and of Theorem 4 in [2]. Now, from a well-known property of the Kuratowski limit it follows that

$$\text{Lim} \int F_n(t)\, d\mu(t) = \text{Lim cl} \int F_n(t)\, d\mu(t) = \int F(t)\, d\mu(t).$$

In other words, we have proved that Kuratowski convergence strictly implies the convergence we introduced in Definition 2.4, while the conclusion of Corollary 3.6 is equivalent to that of Theorem 5 due to Aumann [2].

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